Analytical Series Expressions for the Self- and Mutual Inductances of Two-Dimensional Coils in the Form of Partial Sectors

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Pulsed power generators of the drum type utilize radial components of magnetic fields created by field coils located on a rotor to induce voltage on stator coils located at larger radii. Conventionally, two-dimensional field analyses are used, and further, the field is expressed as a harmonic series and the fundamental part of the radial field is used for initial designs. Analytical series expressions for two-dimensional potentials and fields are utilized in this paper to derive analytical series expressions for the self-inductances and mutual inductances between the rotor and stator coils in the shape of partial sectors. The magnitudes of higher order terms are compared with that of the fundamental component. The effects of parameters such as the aspect ratios of conductor sections on the inductances and coupling constants are evaluated.

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Venkataraman THIAGARAJAN

Abstract--**Pulsed power generators of the drum type utilize radial components of magnetic fields created by field coils located on a rotor to induce voltage on stator coils located at larger radii. Conventionally, two-dimensional field analyses are used, and further, the field is expressed as a harmonic series and the fundamental part of the radial field is used for initial designs. Analytical series expressions for two-dimensional potentials and fields are utilized in this paper to derive analytical series expressions for the self-inductances and mutual inductances between the rotor and stator coils in the shape of partial sectors. The magnitudes of higher order terms are compared with that of the fundamental component. The effects of parameters such as the aspect ratios of conductor sections on the inductances and coupling constants are evaluated.**

I. INTRODUCTION

PULSED power generators are conventionally of two types. The disk type generators utilize axial fields to generate voltage, while the drum type utilize radial fields for the same purpose. The magnetic energy stored, due to currents flowing in the field coils usually located on the rotor, is transferred to the load through currents induced in the stator coils. The process of storing energy and induction of load currents may be described by the self- and mutual inductances of the coils and the coupling constants which are functions of the inductances. Generally, three-dimensional magnetic field calculations are required for the computation of inductances. Two-dimensional calculations might suffice for initial designs of the drum type, since the fields are approximately two-dimensional in the central portions of the machine. Further, the fields could be expanded in harmonic series, and initial designs normally use just the fundamental component of the magnetic field to arrive at optimal designs. The locations of the conductors and their aspect ratios will affect the inductances and magnitudes of the higher order fields.

Methods for the calculation of inductances exist in literature [1]-[4]. Hughes and Miller [2], Bird and Woodson [3], and Parekh [4] address the inductances of two-dimensional coil sections of interest (partial sectors) in pulsed power generators utilizing radial fields for generation. Hughes and Miller [2] derived expressions for the inductances of partial sectors. They assumed sinusoidal current sheet sources and truncated series expressions (with just ten coefficients) for fields consistent with the solution of Laplace's equation and evaluated the coefficients with boundary and interface conditions. Their results presented in a tabular form are not readily adaptable for parametric analyses. Bird and Woodson [3] and Parekh [4] have presented expressions for similar inductances which are amenable to parametric evaluations. But they have made the following simplifying assumptions in their analyses, specific for their application, limiting general validity: (1) the source is described by current sheets with sinusoidal distribution, (2) the azimuthal magnetic field has been assumed to be zero at the radius of the rotor, and (3) the radial magnetic field has been assumed to be zero at the outer radius of the stator.

In this paper, drum type generators with coil sections in the form of partial sectors bounded by given radii and azimuthal angles have been analyzed in two dimensions. The magnetic energy stored is calculated using an integral of the dot product of the current density and the vector potential over the finite sections of the conductors. Two-dimensional series fields have been utilized to derive analytically the self- and mutual inductances and the coupling constants as converging series. No simplifying assumptions and series truncations have been made, and exact analytical series expressions have been derived. Sinusoidal distribution of sources was an essential input for the analyses in the works described earlier. The expressions presented in this paper are valid for general distribution of source conductors (symmetric, asymmetric, or sinusoidal). The expressions for the inductances presented are in a form suitable for parametric analysis, evaluation of the effect of aspect ratios and conductor placement, etc., in initial two-dimensional designs. The inductances and coupling constants have been evaluated as functions of the coil locations and aspect ratios of coil sections. The effects of higher order terms on the inductances are evaluated.

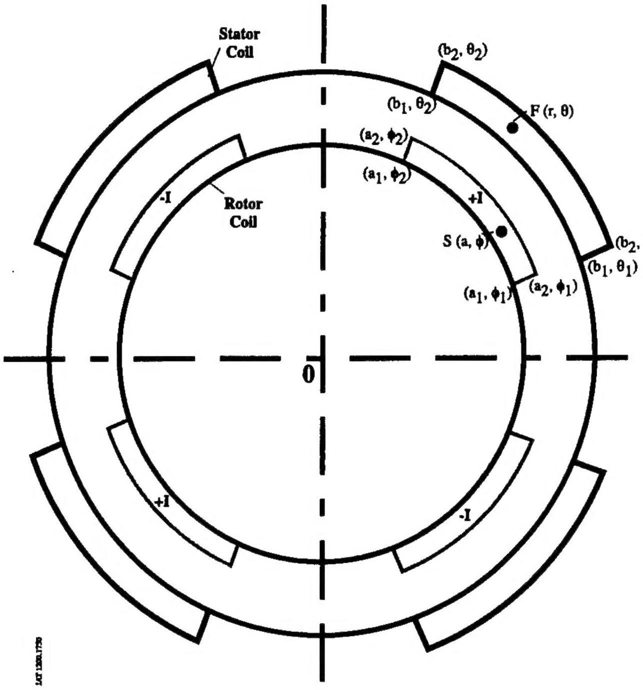

Fig. 1. Schematic coil sections.

II. DESCRIPTION OF THE PROBLEM

A typical drum type generator is shown Fig. 1. The rotor is of radius a_1, and the stator is located at a larger radius b_1. The rotor coils are bounded by radii a_1 and a_2. The coil in the first quadrant varies from ϕ_1 to ϕ_2. The data for the coils in the other regions may be specified by similar data. If the arrangement is symmetric, the azimuthal spreads of the rotor coils in the other quadrants can be obtained utilizing symmetry; otherwise, they will have to be specified as data. Stator coils are bounded by radii b_1 and b_2. The coil in the first quadrant is bounded by angles θ_1 and θ_2. As before, the data for the coils in the other quadrants can be specified individually for general cases and obtained using symmetry, in symmetric arrangements. The configuration shown is a quadrupole with currents flowing in opposite directions in successive quadrants. The data may be specified similarly for higher order poles. The objective is to derive analytical convergent series expressions for the self-inductances, mutual inductances, and the coupling constants as functions of the radii, the azimuthal angles, and the number of poles. The series will be used to analyze parametrically the effects of aspect ratios and radial spreads on the inductances. The analytical expressions may easily be generalized to configurations with many layers of coils and conductors divided into sub-sections with intervening insulators.

III. GOVERNING EQUATIONS

We will use conventional electromagnetic notation. The energy stored in the magnetic field set up by a system of current loops is given by

$$W_m = \frac{1}{2} \int_v B \bullet H \, \mathrm{dv}. \tag{1}$$

This expression for energy may also be written as follows using the vector potential A [5]:

$$W_m = \frac{1}{2} \int_v J \bullet A \, \mathrm{dv} + \frac{1}{2} \int_s A \times H \bullet \mathrm{ds}. \tag{2}$$

The first integral in (2) will be restricted to the volume of the conductors, since the current density elsewhere will be zero. The second integral over the bounding surface will reduce to zero as the domain extends to infinity. The total energy may be obtained using (1) with an infinite domain, or using (2) with the integration domain restricted to those of the current carrying conductors. Equation (2) simplifies to:

$$W_m = \frac{1}{2} \int_v J \bullet A \, \mathrm{dv}. \tag{3}$$

The integral in (3) will reduce to integration in two dimensions (r,θ), considering unit depth along the z direction. Two-dimensional fields can be described by just the z component of the potential A.

$$\vec{A} = \hat{e}_z \, A_z \, ; \;\; B = \nabla \times A \tag{4}$$

The current densities are considered uniform here. Using Fig. 1, the potential A_z at a field point (r,θ) due to a current source $I = j_r.a.da.d\phi$ located at the source point (a,ϕ) may be written down using Ampere's circuital law. This expression for the potential may be expanded in a convergent series, using logarithmic expansions [6]-[8]. Two cases arise, depending on whether the field radius, r, is greater or less than the source radius, a. These two series may be integrated term by term over the source radius a, varying from a_1 to a_2 and angles ϕ_1 to ϕ_2, to obtain the potential due to a single coil section. The fields due to other coil sections may be calculated with limits on the radii and the angles specified individually for asymmetric cases, or, simplified in symmetric cases. The stored magnetic energy may then be calculated using (3). The complete derivation is shown in appendices A and B. The integrated results from these appendices will be used in the following sections with appropriate integration limits.

IV. EXPRESSIONS FOR THE INDUCTANCES

A generator with p number of poles and single layers of coils in the rotor and stator will be considered in the following derivation. A four-pole machine is shown in Fig. 1. The results may easily be extended to more complicated cases by extending the summations. The total vector potential at the field point (r,θ) with sources located between radii a_2 and a_1, and azimuthal spread ϕ_2 and ϕ_1, can be written down using equations B.4 and B.5 as follows, after applying the limits a_2

and a_1 on the source radius a, and ϕ_2 and ϕ_1 on the angle ϕ. The angles of displacement of coil sections (referred to the first section), α_m, will be specified by the design in asymmetric cases or by mirror symmetry ((5) below). If the individual coil sections are located centrally in successive sectors (total p), then

$$\alpha_m = (m-1)\frac{2\pi}{p} \tag{5}$$

$$A_z = \frac{\mu_0 j_r}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n^2 r^n (n+2)}[a_2^{n+2} - a_1^{n+2}]* \\ \sum_{m=1}^{p}(-1)^{m+1}[\sin n(\alpha_m + \phi_2 - \theta) \\ -\sin n(\alpha_m + \phi_1 - \theta)] \quad \text{for } r > a_2 > a_1 \tag{6}$$

$$A_z = \frac{\mu_0 j_r}{2\pi}\sum_{n=1}^{\infty}\frac{r^n}{n^2}\{\frac{(a_2^{-n+2} - a_1^{-n+2})}{(-n+2)}\}* \\ \sum_{m=1}^{p}(-1)^{m+1}[\sin n(\alpha_m + \phi_2 - \theta) \\ -\sin n(\alpha_m + \phi_1 - \theta)] \quad \text{for } r < a_1 < a_2 \tag{7}$$

The current density, j, has been denoted with a subscript r, since the source is located on the rotor in our case. The term inside the curly parentheses in (7) becomes indeterminate for n=2 and may be evaluated using L'Hospital's rule to be $[\ln(a_1/a_2)]$. The fields B_r and B_θ at any point may be obtained using (4).

A. *Mutual Inductance*

Derivation of the series expression for the mutual inductance is straightforward and will be addressed first. Referring to Fig. 1, the stator coil is located between radii b_2 and b_1 ($b_2>b_1>a_2>a_1$) and has an azimuthal spread θ_2 to θ_1 ($\theta_2>\theta_1$). The energy stored can be calculated using (3) and (6). The energy corresponding to the mutual induction between the rotor coils and the stator coil section in the first quadrant (Fig. 1) carrying a current density, j_s, can be obtained by integrating (3) over the section of the stator coil with $J=j_s$ and the potential A given by (6). The current density in the stator coil has been denoted with a subscript s to distinguish it from the current density in the rotor coils, j_r. The limits for integration on r will be b_1 and b_2, and those on θ will be θ_1 and θ_2. The result is,

$$W_m = \frac{\mu_0}{4\pi} j_r j_s \sum_{n=1}^{\infty}\frac{1}{n^3}[\frac{(a_2^{n+2} - a_1^{n+2})}{(n+2)}]*[\frac{(b_2^{-n+2} - b_1^{-n+2})}{(-n+2)}]* \\ \sum_{1}^{p}(-1)^{m+1}[\cos n(\alpha_m + \phi_2 - \theta_2) - \cos n(\alpha_m + \phi_2 - \theta_1) \\ -\cos n(\alpha_m + \phi_1 - \theta_2) + \cos n(\alpha_m + \phi_1 - \theta_1)] \tag{8}$$

A similar expression may be derived using (7) for cases where the source radii are larger than the field radii. The mutual inductance can be related to the energy stored through the following equation [5].

$$W_m = \frac{1}{2} M\, I_r\, I_s \tag{9}$$

The mutual inductance is indicated by M, and the I's refer to the currents. The right hand side of (9) would be multiplied by 2 if the energy on the left hand side were to include the mutual inductive energy two ways, rotor to stator and stator to rotor. The currents are expressed as follows in terms of the current densities and sectional areas:

$$I_r = j_r(a_2{}^2 - a_1{}^2)(\phi_2 - \phi_1)/2 \tag{10a}$$

$$I_s = j_s(b_2{}^2 - b_1{}^2)(\theta_2 - \theta_1)/2. \tag{10b}$$

Using (8) to (10), we derive the expression for the mutual inductance (in Henries/m) to be

$$M = \frac{\mu_0}{2\pi}\left[\frac{2}{(b_2^2 - b_1^2)(\theta_2 - \theta_1)}\right]\left[\frac{2}{(a_2^2 - a_1^2)(\phi_2 - \phi_1)}\right]* \\ \sum_{n=1}^{\infty}\frac{1}{n^3}\left[\frac{(a_2^{n+2} - a_1^{n+2})}{(n+2)}]*[\frac{(b_2^{-n+2} - b_1^{-n+2})}{(-n+2)}\right]* \\ \sum_{m=1}^{p}(-1)^{m+1}[\cos n(\alpha_m + \phi_2 - \theta_2) - \cos n(\alpha_m + \phi_2 - \theta_1) \\ -\cos n(\alpha_m + \phi_1 - \theta_2) + \cos n(\alpha_m + \phi_1 - \theta_1)]. \tag{11}$$

The second term in square brackets after the first summation in (8) or (11) becomes indeterminate for n=2. Applying L'Hospital's rule, it is found to be $[-\ln(b_1/b_2)]$. It should be noted that (11) gives the mutual inductance for just one stator coil section. For the case shown in Fig. 1, this value should be multiplied by 4 for all the sections. If there is no symmetry, (11) will have to be applied for every section with appropriate limits on the radius, b, and azimuthal angle, θ, and the results summed up to obtain the net mutual inductance. Equation (11) is a general expression for the mutual inductance, M, as a function of the radii and azimuthal angles. If the stator coils are arranged in more than one radial layer, additional terms should be included.

B. *Self-Inductance*

The potentials or fields for radii bounded by the inner and outer radii of the rotor conductors, i.e., in the region $a_2>r>a_1$, are needed for the evaluation of the self-inductances. These will have to be obtained differently, as shown in Appendix B, using equations B.4 and B.5. This region can be divided into sub-regions for the evaluation of the vector potential: the first region bounded by a_1 and r, and the second region bounded by r and a_2. The potential at (r,θ) will be given by the sum of: (1) potentials contributed by sources between a_1 and r, and (2)

potentials contributed by sources between r and a_2. The resulting expression for A_z is

$$A_z = \frac{\mu_0 j_r}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \{ [\frac{1}{n+2}(r^2 - \frac{a_1^{n+2}}{r^n})] + [\frac{1}{-n+2}(a_2^{-n+2} r^n - r^2)] \} * \sum_{m=1}^{p} (-1)^{m+1} [\sin n(\alpha_m + \phi_2 - \theta) - \sin n(\alpha_m + \phi_1 - \theta)]. \tag{12}$$

As before, α_m is given by (5) for symmetric cases, and will have to be specified as data for general cases. The second term within the curly parentheses in (12) will become indeterminate for n=2. This can be determined by applying L'Hospital's rule to be $= -r^2 \cdot \ln(r/a_2)$. The energy stored, W_s, through self-induction can be obtained by using (3) and (12). The I will be given by (j_r.r.dr.dθ), and the limits on r and θ will be a_1 to a_2 and θ_1 to θ_2.

$$W_s = \frac{\mu_0 j_r^2}{4\pi} \sum_{n=1}^{\infty} \frac{1}{n^3} [\frac{(a_2^4 - a_1^4)}{4(n+2)} - \frac{a_1^{n+2}(a_2^{-n+2} - a_1^{-n+2})}{(n+2)(-n+2)} + \frac{a_2^{-n+2}(a_2^{n+2} - a_1^{n+2})}{(n+2)(-n+2)} - \frac{(a_2^4 - a_1^4)}{4(-n+2)}] * \sum_{m=1}^{p} (-1)^{m+1} [\cos n(\alpha_m + \phi_2 - \theta_2) - \cos n(\alpha_m + \phi_2 - \theta_1) - \cos n(\alpha_m + \phi_1 - \theta_2) + \cos n(\alpha_m + \phi_1 - \theta_1)]. \tag{13}$$

The limits θ_2,θ_1 will be equal to ϕ_2,ϕ_1 for the first coil section. The value obtained from (13) should be multiplied by 4 for the case shown in Fig. 1. If there is no symmetry, (13) should be applied to every coil section with appropriate limits and the values for W_s summed. When n=2, the terms within the first square brackets should be replaced by

$$[\frac{(a_2^4 - a_1^4)}{4(n+2)} + \frac{a_1^{n+2}}{(n+2)} \ln(\frac{a_1}{a_2}) + \frac{(a_2^4 - a_1^4)}{16} - \frac{(a_2^4 \ln(a_2) - a_1^4 \ln(a_1))}{4} + \frac{(a_2^4 - a_1^4)}{4} \ln(\frac{a_1}{a_2})] \tag{13a}$$

The energy stored through self-induction can also be expressed as [5]

$$W_s = \frac{1}{2} L I_r^2, \tag{14}$$

where the current, I_r, is given by (10a). Therefore, the expression for the self-inductance, L, of a coil with inner and outer radii, a_1,a_2, and azimuthal spread, ϕ_1,ϕ_2, and with an angular displacement of coils, α_m, between poles becomes

$$L = \frac{\mu_0}{4\pi} \frac{8}{(a_2^2 - a_1^2)^2 (\phi_2 - \phi_1)^2} \sum_{n=1}^{\infty} \frac{1}{n^3} [\frac{(a_2^4 - a_1^4)}{4(n+2)} - \frac{a_1^{n+2}(a_2^{-n+2} - a_1^{-n+2})}{(n+2)(-n+2)} + \frac{a_2^{-n+2}(a_2^{n+2} - a_1^{n+2})}{(n+2)(-n+2)} - \frac{(a_2^4 - a_1^4)}{4(-n+2)}] * \sum_{m=1}^{p} (-1)^{m+1} [\cos n(\alpha_m + \phi_2 - \theta_2) - \cos n(\alpha_m + \phi_2 - \theta_1) - \cos n(\alpha_m + \phi_1 - \theta_2) + \cos n(\alpha_m + \phi_1 - \theta_1)]. \tag{15}$$

The expression within the first square brackets should be replaced by (13a) for n=2. It can be shown easily [7] that for a symmetric dipole, terms with n=1,3,5,7...etc. survive, and the rest will be zero; for a symmetric quadrupole, terms with n=2,6,10,14...etc. alone survive; and so on for higher order poles. Equation (15) gives the self-inductance in Henries/m. The coupling constant K is given by [5],

$$K = \frac{M}{\sqrt{(L_r L_s)}}. \tag{16}$$

K may be obtained using (11), (15), and (16). An approximate expression for K may be obtained by using the fundamental components alone—i.e., by restricting the summation to just n=1 for a dipole and n=2 for a quadrupole, etc. It should be noted that the angle of displacement, α_m, will be given by (5) for symmetry and will have to be specified as data for general cases. It could also be different for different layers, the rotor, and the stator.

V. COMPARISON WITH FINITE ELEMENT SOLUTIONS

Equations (11) and (15) give analytical expressions for the mutual and self-inductances. These may also be calculated numerically using finite elements or other numerical procedures. The code Opera2d [8] was used to compute the fields and stored energy for some sample cases. One quadrant of a quadrupole configuration (Fig. 1) was analyzed with Opera2d with a coil with an azimuthal spread of 10° to 80°, inner radius of 0.3 m, and outer radii varying from 0.32 m to 0.44 m. The current density input was 1.0E7 A/m^2. The outer boundary was set at a radius of 3.0 m. The fields were analyzed and the energy stored per quadrant computed using Opera2d and also (13). The results are shown plotted in Fig. 2. The differences between the computed and analytical results are found to be negligible.

As a second example, two coil sections in a quadrant of a quadrupole configuration were analyzed. The inner coil was set with a_1=0.3 m, a_2=0.34 m, ϕ_1=10°, and ϕ_2=80°. The outer coil was set with b_1=0.38 m, b_2=0.46 m, θ_1=0°, and θ_2=50°. The current density in the coils was set at 1.0E7 A/m^2. Five different cases were analyzed with the outer coil rotated by 10° for every case. For each case, three different analyses were carried out: (1) inner coil energized (energy W_1), (2) outer coil energized (energy W_2), and (3) both coils energized (energy

W_3). W_1,W_2 correspond to the self inductances of the inner and outer coils respectively. The value $W_m = (W_3\text{-}W_1\text{-}W_2)/2$ corresponds to the mutual inductance between the coils [5]. The value of W_1 = 2698.5 J/m, computed using Opera2d, matches the value 2698.0 J/m computed using (13). W_2 and W_m vary with angle. W_m is shown plotted in Fig. 3 and compared with analytical results from (8). The maximum difference is found to be less than 4%, which is the accuracy achievable with finite element solutions.

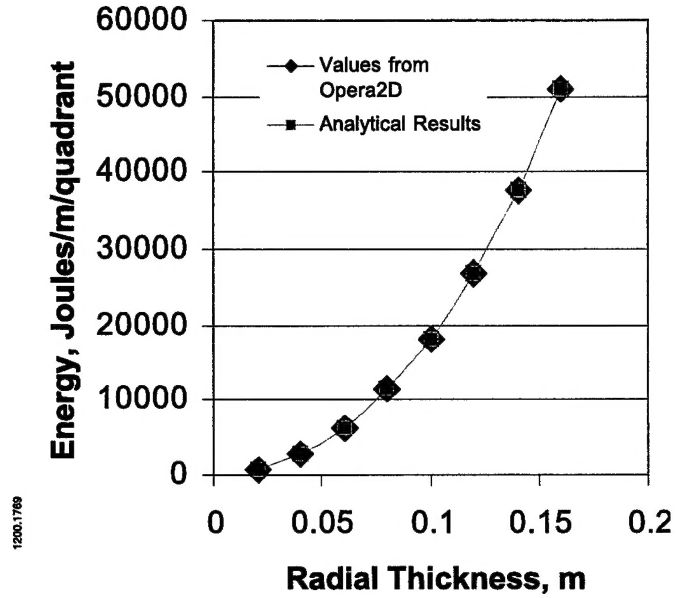

Fig. 2. Comparison of energies stored through self-induction.

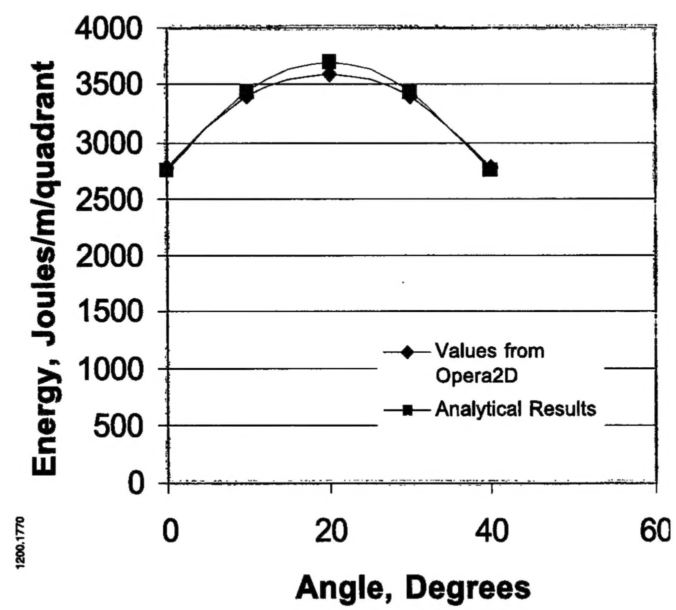

Fig. 3. Comparison of energies stored through mutual induction.

VI. PARAMETRIC ANALYSES

Equations (11) and (15) are analytical expressions for the inductances as functions of the geometrical parameters. They may be used to optimize design parameters parametrically. As an example, the quadrupole configuration shown in Fig. 1 has been analyzed varying the dimensions of the coil. The radial thickness of the rotor coil was varied, and the azimuthal spread of the coil was reduced, while keeping the sectional area constant. The dimensions of the stator coil were maintained constant. The increase in the radial thickness of the rotor coil divided by the initial radial thickness is described as the normalized aspect ratio (or just "aspect ratio") in the following. The self- and mutual inductances and the coupling constants were computed for aspect ratios ranging from 1 to 1.1. The inductances are shown plotted in Fig. 4 and the coupling constants are shown in Fig. 5 plotted against the aspect ratios. The self-inductance of the rotor coils is found to increase from 0.68 to 1.12 μH, while the mutual inductance drops from 0.44 to 0.27 μH. The self-inductance of the stator coil remains constant at 0.68, since its section is maintained constant. The resulting coupling constant decreases from 0.64 to 0.31 when the aspect ratio of the rotor coil is varied from 1 to 1.1.

In a similar fashion, the section of the rotor coil was maintained constant and the radial thickness of the stator coil was varied, keeping the sectional area constant. The computed results are shown in Figs. 6 and 7. The coupling constant with a value of 0.64 at an aspect ratio of 1.0 increases slightly to 0.66 and then decreases to a value of 0.52 at an aspect ratio of 1.1. The inductances and coupling constants are predominantly governed by the fundamental component of the symmetric quadrupole. The second harmonic with n=6 decreases by a factor of about 27 when compared with the fundamental.

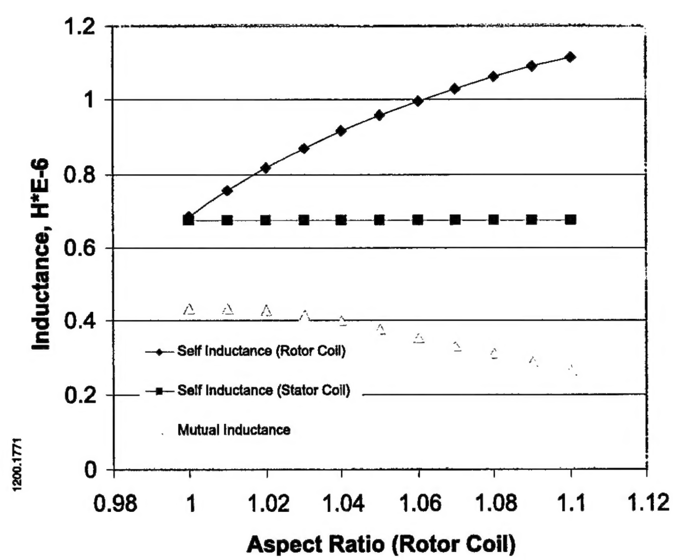

Fig. 4. Inductance vs. aspect ratio (rotor coil).

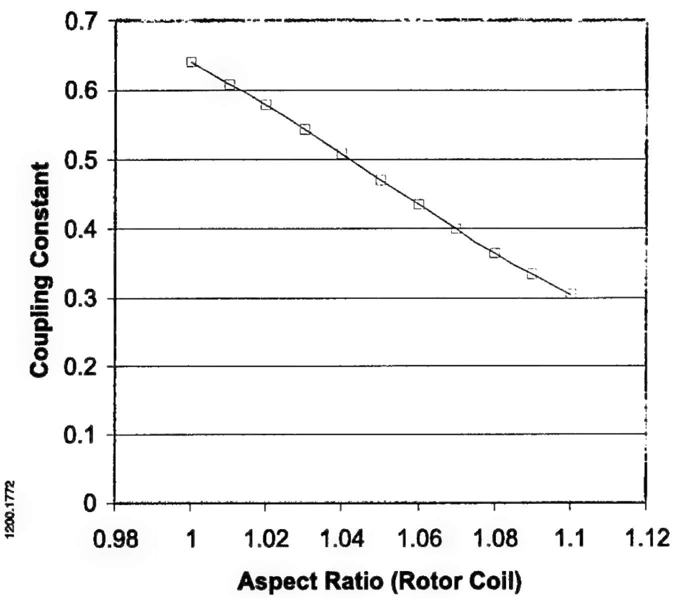

Fig. 5. Coupling constant vs. aspect ratio of rotor coil.

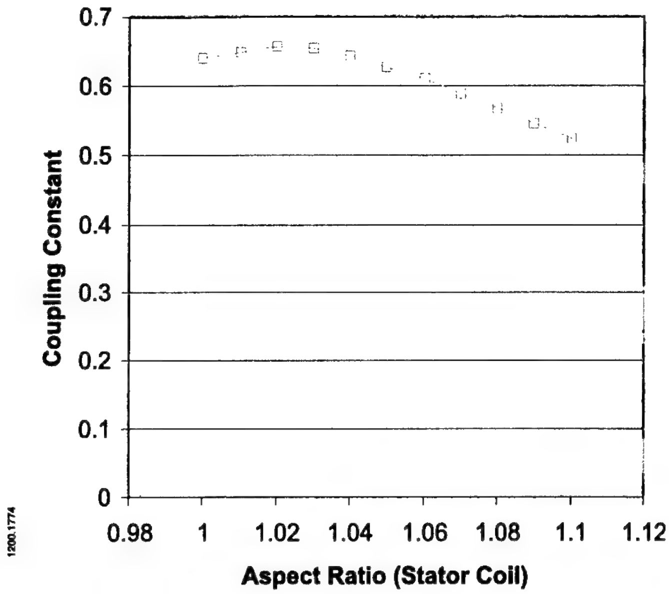

Fig. 7. Coupling constant vs. aspect ratio (stator coil).

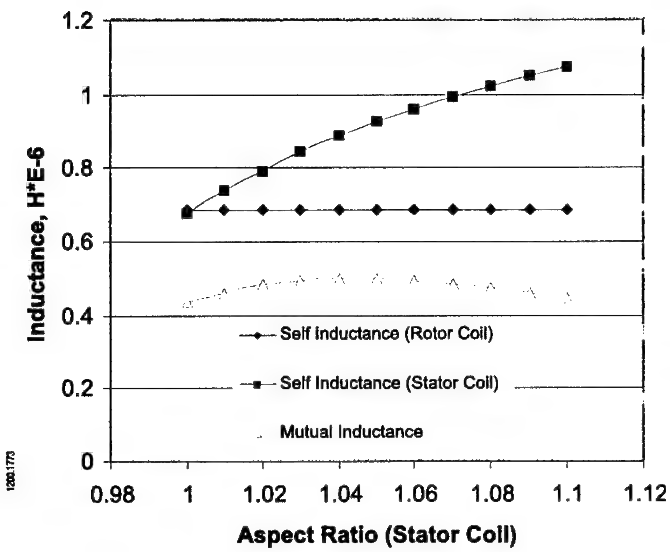

Fig. 6. Inductance vs. aspect ratio (stator coil).

VII. SUMMARY

Analytical series expressions for the self-inductances and mutual inductances as a function of the azimuthal angle have been derived for two-dimensional coils in the form of partial sectors located on circular peripheries in drum configurations. The values calculated with the present expressions have been checked with two-dimensional finite element computations. These expressions will be useful in initial parametric designs and the assessment of the effects of harmonics in drum type alternators.

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APPENDIX A

The formulae for the two-dimensional potentials and fields created by conductors in the form of partial sectors bounded by inner and outer radii a_1, a_2, and angles ϕ_1, ϕ_2, have been derived in this appendix. The magnetic field B is related to the vector potential A by the following equation:

$$\vec{B} = \nabla \times \vec{A} = \hat{e}_r \frac{1}{r}\frac{\partial A_z}{\partial \theta} + \hat{e}_\theta\left(\frac{-\partial A_z}{\partial r}\right) \tag{A.1}$$

Two-dimensional fields can be described by just the z component of the potential A.

$$\vec{A} = \hat{e}_z\, A_z \tag{A.2}$$

The field B_θ at a point $F(r,\theta)$ around a conductor located at a point $S(a,\phi)$ and carrying a differential current = I = j.a.da.dϕ is given by the following equation. Let the radial co-ordinate from the source point S be R. Then,

$$B_\theta \;=\; \frac{\mu_0 I}{2\pi R} = -\frac{\partial A_z}{\partial R} \tag{A.3}$$

The following expression for the potential can be obtained by integrating once over R.

$$A_z = \frac{-\mu_0 I}{2\pi}\ln R = \frac{-\mu_0 \mathrm{j}\ \mathrm{a\ da\ d}\phi}{2\pi}\ln R \tag{A.4}$$

The radial distance R can be expressed in terms of the global source and field co-ordinates as follows.

$$R^2 = r^2 + a^2 - 2ar\cos(\phi - \theta) \tag{A.5}$$

Therefore, the expression for the potential can be written as follows.

$$A_z = \frac{-\mu_0 I}{4\pi}\ln(r^2 + a^2 - 2ar\cos(\phi - \theta)) \tag{A.6}$$

The current I could be replaced by j.a.da.dϕ for further integrations (j being the current density). The components of the magnetic fields could be obtained using A.6 and A.1.

The logarithmic expression in A.6 could be expanded in series and two cases arise depending on whether the field radial co-ordinate r is greater or less than the source radial co-ordinate a.

Case I (r > a):

A.6 can be re-written for this case as follows:

$$A_z = \frac{-\mu_0 I}{4\pi}\ln\left\{r^2\left[1 + \frac{a^2}{r^2} - \frac{2a}{r}\cos(\phi - \theta)\right]\right\}$$

The expression within the curly parenthesis may be re-written as follows with $i = \sqrt{-1}$ [1,2].

$$A_z = \frac{-\mu_0 I}{2\pi}\ln(r) - \frac{\mu_0 I}{4\pi}\ln\ \left\{\left[1 - \frac{a}{r}\exp(i(\phi-\theta))\right]*\left[1 - \frac{a}{r}\exp(-i(\phi-\theta))\right]\right\} \tag{A.7}$$

The logarithmic expressions may be expanded using the series for ln(1-x) and A.7 may be expanded as follows using the de Moivre's theorem.

$$A_z = \frac{-\mu_0 I}{2\pi}\ln(r) + \frac{\mu_0 I}{2\pi}\left[\frac{a}{r}\cos(\phi-\theta) + \frac{1}{2}\frac{a^2}{r^2}\cos 2(\phi\text{-}\theta)\ldots + \frac{1}{n}\frac{a^n}{r^n}\cos n(\phi-\theta) + \ldots\right] \tag{A.8}$$

The series expressions for the field components B_r, B_θ follow using A.1

$$B_r = \frac{\mu_0 I}{2\pi}\left\{\frac{a}{r^2}\sin(\phi-\theta) + \frac{a^2}{r^3}\sin 2(\phi-\theta) + \ldots. + \frac{a^n}{r^{n+1}}\sin n(\phi-\theta) + \ldots\right\} \tag{A.9}$$

$$B_\theta = \frac{\mu_0 I}{2\pi r} + \frac{\mu_0 I}{2\pi}\left\{\frac{a}{r^2}\cos(\phi-\theta) + \frac{a^2}{r^3}\cos 2(\phi-\theta) + \ldots. + \frac{a^n}{r^{n+1}}\cos n(\phi-\theta) + \ldots\right\} \tag{A.10}$$

Case II (r<a):

A.6 can be written as follows for this case:

$$A_z = \frac{-\mu_0 I}{4\pi}\ln\left\{a^2\left[1 + \frac{r^2}{a^2} - \frac{2r}{a}\cos(\phi-\theta)\right]\right\}$$

The above equation may be re-written as:

$$A_z = \frac{-\mu_0 I}{2\pi}\ln(a) - \frac{\mu_0 I}{4\pi}\ln\left\{\left[1 - \frac{r}{a}\exp(i(\phi-\theta))\right] * \left[1 - \frac{r}{a}\exp(-i(\phi-\theta))\right]\right\} \tag{A.11}$$

As in the previous case, the logarithmic expressions may be expanded in series and de Moivre's theorem applied to get:

$$A_z = \frac{-\mu_0 I}{2\pi}\ln(a) + \frac{\mu_0 I}{2\pi}\left[\frac{r}{a}\cos(\phi-\theta) + \frac{1}{2}\frac{r^2}{a^2}\cos 2(\phi\text{-}\theta) + \ldots. + \frac{1}{n}\frac{r^n}{a^n}\cos n(\phi-\theta) + \ldots.\right] \tag{A.12}$$

The series expressions for the field components B_r, B_θ are obtained using A.1

$$B_r = \frac{\mu_0 I}{2\pi}\left\{\frac{1}{a}\sin(\phi-\theta) + \frac{r}{a^2}\sin 2(\phi-\theta) + \ldots. + \frac{r^{n-1}}{a^n}\sin n(\phi-\theta) + \ldots\right\} \tag{A.13}$$

$$B_\theta = -\frac{\mu_0 I}{2\pi}\left\{\frac{1}{a}\cos(\phi-\theta) + \frac{r}{a^2}\cos 2(\phi-\theta) + \ldots. + \frac{r^{n-1}}{a^n}\cos n(\phi-\theta) + \ldots\right\} \tag{A.14}$$

APPENDIX B

The energy stored in the magnetic field set up by a system of current loops is given by

$$W_m = \frac{1}{2}\int_v B \bullet H \text{ dv} \tag{B.1}$$

This expression for energy may also be written as follows using the vector potential A [3]:

$$W_m = \frac{1}{2}\int_v J \bullet A \text{ dv} + \frac{1}{2}\int_s A \times H \bullet \text{ds} \tag{B.2}$$

The first integral in B.2 will be restricted to the volume of the conductors, since the current density elsewhere will be zero. The second integral over the bounding surface will reduce to zero as the domain extends to infinity. The total energy may be obtained using B.1 with an infinite domain, or using B.2 with the integration domain restricted to those of the current carrying conductors. Equation B.2 simplifies to:

$$W_m = \frac{1}{2}\int_v J \bullet A \text{ dv} \tag{B.3}$$

The current densities in the conductors treated in this paper will be uniform, but could have different values for the source and field conductors. They will be denoted by j_r, j_s respectively. The energy will be obtained per unit length of the conductors, and therefore the integration will be carried out with dv=a.da.dϕ. As seen from equations A.8 and A.12, the vectors J and A are parallel. The integration domain can be divided into two regions: region I with the field point radius r > a (source point radius) and region II with r < a. A_z for region I can be obtained as follows using equation A.8. The first term will be ignored, since pairs of conductors with opposing currents will be considered. The current I will be replaced by j_r.a.da.dϕ. The integral is

$$\begin{aligned} A_z &= \iint \frac{\mu_0 j_r}{2\pi} \left\{ \sum_{n=1}^{\infty} \frac{1}{n} \frac{a^n}{r^n} \cos n(\phi - \theta) \right\} a \text{ da d}\phi \\ &= \frac{\mu_0 j_r}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1}{r^n} \frac{a^{n+2}}{(n+2)} \sin n(\phi - \theta) \end{aligned} \tag{B.4}$$

In region II where r < a, an equation for A_z may be derived in a similar fashion. The result is

$$\begin{aligned} A_z &= \iint \frac{\mu_0 j_r}{2\pi} \left\{ \sum_{n=1}^{\infty} \frac{1}{n} \frac{r^n}{a^n} \cos n(\phi - \theta) \right\} a \text{ da d}\phi \\ &= \frac{\mu_0 j_r}{2\pi} \sum_{n=1}^{\infty} \frac{r^n}{n^2} \frac{a^{-n+2}}{(-n+2)} \sin n(\phi - \theta) \end{aligned} \tag{B.5}$$

Appropriate limits on the radial coordinate a and the angular coordinate ϕ should be applied in equations B.4 and B.5, depending on the spatial extent of the source coils. When the coil consists of more than one section, a summation over the trigonometric terms should be included in equations. B.4 and B.5. The resulting equations will yield the potential A_z as functions of the field coordinates r and θ. These may be substituted in equation B.3 and integrated over the spatial domain of the field conductors to obtain the magnetic energy stored.

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